A derivation of the maximum principle for control problems with a discontinuous system of equations is given in Ref. 6, pages 311-312. The optimality condition requires

$$u(t) = -\operatorname{sgn}\lambda_1(t) \tag{21}$$

Thus, the solution is bang-bang-type control of E's commanded acceleration similarly to the ideal system case,[4] where its actual acceleration was bang-bang.

Computational Results

The solutions for the problem were obtained numerically using a multiple shooting algorithm. The initial states are all zero and the following numerical values are used:

$$N' = 3, \qquad V_p' = 1000 \text{ fps}, \qquad V_e' = 500 \text{ fps}$$

$$\dot{\gamma}_{e_m} = 0.3 \text{ s}^{-1}, \qquad \tau_p = 0.5 \text{ s}, \qquad t_f = 3.75 \text{ s}$$

A two-parameter family of solutions has been obtained by varying $(\ddot{y}_{pm}/\ddot{y}_{em})$ and (τ_e/τ_p) (i.e., the relative maneuverability and the relative time response of the opponents are used as parameters). The miss-distance results are presented as a function of these parameters in Fig. 2. Note that E's time constant need not be smaller than P's for evading with a significant miss distance, and that a nonzero miss distance is guaranteed even when the pursuer has unlimited maneuverability.

The problem as formulated has a closed-form solution for $\tau_e = 0$, $\ddot{y}_{p_m} = \infty$, which has been obtained in Ref. 4. For this case, E applies bang-bang-type control with one switching point at $\theta \equiv (t_f - t)/\tau_p = 2$. For other values of τ_e and $\ddot{y}_{pm}$, the variable θ is shown in Fig. 3. In all of these results, the guidance parameter N' is 3.

In Ref. 4, E's system was approximated by a ramp function with t_r as the ramp time (the minimum time to change the lateral acceleration from $-\ddot{y}_{e_m}$ to $\ddot{y}_{e_m}$). Their results may be compared to the present results by approximating $t_r = 3\tau_e$. In general, the comparison is satisfactory for both the miss distance and the switching point. For a "slow" evader $(\tau_e/\tau_p > 1)$ but with relatively high maneuverability $(\ddot{y}_{e_m}/\ddot{y}_{pm} > 0.25)$, the ramp time approximation results are more optimistic (for E) and the predicted miss distances are greater than these of the first-order model. For a "fast" evader or for a less maneuverable one, the results are in very good agreement.

Concluding Remarks

By applying linearized kinematics to the optimal evasion problem, the optimal commanded lateral acceleration was found to be a "bang-bang" nonsingular control governed by a switching function. The optimal miss distance is dependent on the relative maneuverability of the pursuer and the evader, and on the relative time response. However, the evader need not be faster in its dynamic responses in order to guarantee a finite miss distance even in conflict with a pursuer of unlimited maneuverability.

Acknowledgment

This research was supported in part by the Defense Advanced Research Projects Agency under ACMP Grant F 49620-87-C-0016.

References

[1]Isaacs, R., *Differential Games*, Wiley, New York, 1965.

[2]Julich, P. M. and Borg, D. A., "Proportional Navigation vs an Optimally Evading Constant-Speed Target in Two Dimensions," *Journal of Spacecraft and Rockets*, Vol. 7, Dec. 1970, pp. 1454-1457.

[3]Slater, G. L. and Wells, W. R., "Optimal Evasive Tactics with Time Delay," *Journal of Spacecraft and Rockets*, Vol. 10, May 1973, pp. 309-313.

[4]Shinar, J. and Steinberg, D., "Analysis of Optimal Evasive Maneuvers Based on a Linearized Two-Dimensional Model," *Journal of Aircraft*, Vol. 14, Aug. 1977, pp. 795-802.

[5]Shinar, J., Rosenzstein, Y., and Bezner, E., "Analysis of Three-Dimensional Optimal Evasion with Linearized Kinematics," *Journal of Guidance and Control*, Vol. 2, Sept. 1979, pp. 353-360.

[6]Pontryagin, L. S. et al., *The Mathematical Theory of Optimal Processes*, Interscience, New York, 1962.

Analytical Solution of Optimal Trajectory-Shaping Guidance

M. N. Rao*
Defence Research and Development Laboratory, Hyderabad, India

Introduction

RECENTLY, Lin and Tsai[1] have presented a combined mid-course and terminal guidance law design for missiles to achieve range enhancement with excellent intercept performance, and have also arrived at the closed-form solution of a closed-loop nonlinear optimum guidance law for three-dimensional flight for both the mid-course and terminal phase by neglecting some nonlinear terms.

In this Note, an analytical solution that includes terms not considered by the cited authors is presented for the preceding problem. The notation adopted would be the same as in Ref. 1.

Method of Solution

The differential Eq. (44) of Lin and Tsai[1] is used for the study and solved in analytical form. That equation is

$$\frac{dk}{dR} = \left[\left(\frac{k^2}{2} - F^2\right)\sin\sigma + \frac{k+C}{R}\right]\left(1+\sin^2\sigma\right)$$

$$R\,\frac{dk}{dR} = \left[\left(\frac{k^2}{2} - F^2\right)R\sin\sigma + (k+C)\right] + \left[\left(\frac{k^2}{2} - F^2\right)R\sin\sigma + (k+C)\right]\sin^2\sigma \tag{1}$$

Let $\lambda = R\sin\sigma$, then

$$\frac{d\lambda}{dR} = R\cos\sigma\,\frac{d\sigma}{dR} + \sin\sigma = -kR$$

by using Eq. (28) of Ref. 1.

Equation (1) leads to

$$-\frac{d^2\lambda}{dR^2} + \frac{(1+C_2)}{R}\frac{d\lambda}{dR} + F_1\lambda - C_1 = (1+\sin^2\bar{\sigma})\,\frac{k^2}{2}\,\lambda \tag{2}$$

where

$$F_1^2 = F^2 + F^2\sin^2\bar{\sigma}$$

$$C_1 = C\,(1+\sin^2\bar{\sigma})$$

$$C_2 = 1 + \sin^2\bar{\sigma}$$

and $\bar{\sigma}$ is the average value of σ.

Received Aug. 3, 1987; revision received May 18, 1988.

*Head Systems Group, Defence Research and Development Laboratory, Hyderabad, India.

By changing the independent variable from R to T by $F_1R = T$, we get

$$\frac{d^2\lambda}{dT^2} - \frac{(1+C_2)}{T}\frac{d\lambda}{dT} - \lambda + \frac{C_1}{F_1^2} = -\lambda\frac{(1+\sin^2\bar{\sigma})}{F_1^2}\frac{k^2}{2} \tag{3}$$

By neglecting the right-hand side, the solution of Eq. (3) that is obtained is more refined, as certain terms not considered in Ref. 1 are included by their average value to show their effect on the behavior of parameter λ:

$$\frac{d^2\lambda}{dT^2} - \frac{(1+C_2)}{T}\frac{d\lambda}{dT} + \frac{C_1}{F_1^2} - \lambda = 0 \tag{4}$$

By further changing the dependent variable by

$$P = (C_1/F_1^2) - \lambda$$

the differential Eq. (4) leads to

$$\frac{d^2P}{dT^2} - \frac{(1+C_2)}{T}\frac{dP}{dT} - P = 0 \tag{5}$$

The solution of Eq. (5) is now obtained in terms of the modified Bessel function of order μ.

Solution of Differential Equation

Changing the dependent variable P by the transformation

$$P = UT^{n+\frac{1}{2}}$$

$$n = (1+C_2)/2$$

so that U is the new dependent variable, the differential Eq. (5) becomes

$$T^2\frac{d^2U}{dT^2} + T\frac{dU}{dT} - \left[\left(\frac{2n+1}{2}\right)^2 + T^2\right]U = 0 \tag{6}$$

By writing

$$(2n+1)/2 = \mu$$

we get

$$T^2\frac{d^2U}{dT^2} + T\frac{dU}{dT} - (\mu^2+T^2)U = 0 \tag{7}$$

The differential Eq. (7) is the standard Bessel's differential equation, which admits solution as

$$U = AI_\mu(T) + BK_\mu(T) \tag{8}$$

where $I_\mu(T)$ and $K_\mu(T)$ are the modified Bessel functions of order μ and A and B are integration constants.

Hence,

$$P = T^{n+\frac{1}{2}}[AI_\mu(T) + BK_\mu(T)] \tag{9}$$

where

$$\mu = 1 + (C_2/2)$$

Going back to the original variables, namely, $R\sin\sigma$ and R, we obtain

$$R\sin\sigma = (C_1/F_1^2) - (F_1R)^\mu[AI_\mu(F_1R) + BK_\mu(F_1R)] \tag{10}$$

where

$$\mu = 1 + (C_2/2)$$

The constants of integration A and B can be determined with the help of conditions as used in Ref. (1).

The solution as given by Eq. (10) is more realistic when compared with that of the solution given by Eq. (48) of Ref. 1.

The solution as obtained by Lin and Tsai can be obtained easily from that given by Eq. (10) by suppressing the terms not considered by them, namely, the $F_1^2\sin^2\bar{\sigma}$ and $\sin^2\bar{\sigma}$. The solution so obtained will be the modified Bessel function of order 3/2.

Thus, the solution obtained is considered more valuable for further analysis and the determination of optimum control gains K_1 and K_2, which are expressed in terms of normal acceleration of a missile by Eq. (19) of Ref. 1. The gains K_1 and K_2 as obtained by Lin and Tsai, are modified with this solution, and their behavior is going to show the nonlinear effect in trajectory-shaping guidance.

Conclusion

In this Note, an analytical solution of the optimal trajectory-shaping guidance law has been presented. The solution offered results in better estimation of the optimal gains of a closed-loop nonlinear system. The missiles employing the above guidance law can achieve higher range with excellent intercept performances. It would also provide a smooth acceleration–time history leading to better end-performance parameters. The solution offered is in terms of well-known functions.

The analytical solution offered in Ref. 1 can be obtained directly on a particular case of this study. This guidance law finds its application for surface-to-surface, surface-to-air, and air-to-air types of missiles.

Acknowledgment

I would like to thank Dr. I Achyuta Rao, Director, Systems, Defence Research and Development Laboratory, for his kind encouragement and valuable discussions.

Reference

[1]Lin, C. F. and Tsai, L. L. "Analytical Solution of Optimum Trajectory-Shaping Guidance," *Journal of Guidance, Control, and Dynamics,* Vol. 10, Jan.-Feb. 1987, pp. 61–66.

Three-Dimensional Energy-State Extremals in Feedback Form

M. D. Ardema*
Santa Clara University, Santa Clara, California
and
N. Rajan† and L. Yang‡
Sterling Software, Palo Alto, California

Introduction

IN Refs. 1 and 2, we have developed an algorithm for real-time near-optimal, three-dimensional guidance of high-performance aircraft in pursuit-evasion and target-interception

Presented as Paper 87-2316 at the AIAA Guidance, Navigation, and Control Conference, Monterey, CA, Aug. 17–19, 1987; received Nov. 17, 1987; revision received May 12, 1988.

*Professor and Chairman, Department of Mechanical Engineering. Associate Fellow AIAA.
†Member of the Technical Staff. Member AIAA.
‡Member of the Technical Staff.